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On ($k = \frac{3}{4}$) coherent states for the harmonic oscillator

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Abstract. A new construction of Perelomov's generalised coherent states is considered for one-dimensional harmonic oscillators admitting the Heisenberg–Weyl group as invariance Lie group. Exploiting the Niederer *maximal* kinematical invariance group for such physical systems, we deduce further characteristics on the Heisenberg states through the use of the fundamental Perelomov state $|k, k\rangle$ with $k = \frac{3}{4}$. We explicitly get new normalisation factor and measure for the Heisenberg generalised coherent states. The real Lie algebras so(2, 1) $\square$ h(2), so(2, 1) and h(2) play a prominent role in this study.

1. Introduction

The largest kinematical group of the (one-dimensional) quantum harmonic oscillator has been determined by Niederer (Niederer 1972, 1973) as the group denoted by HO(1), isomorphic to SCHR(1). It can be used in order to construct coherent states with *maximal* symmetry (Beckers and Debergh 1989a) by intensively exploiting its content at the level of the corresponding Lie algebras. Recall that to this kinematical group corresponds the semidirect sum of two Lie algebras, the so-called Heisenberg–Weyl real algebra h(2) and the real orthogonal algebra so(2, 1).

It is well known that the algebra h(2) as well as the non-compact algebra so(2, 1) are fundamental structures for one-dimensional quantum harmonic oscillators. On the one hand we immediately learn from elementary quantum mechanics (Cohen-Tannoudji *et al* 1977, Shankar 1980) the prominent role of h(2) by studying in particular the energy spectrum of the harmonic oscillator (and its eigenfunctions). On the other hand, if dynamical *or* kinematical symmetries are explored, the simple algebra so(2, 1) directly emerges, as quoted by Wybourne (1974) (in the dynamical context) or by Niederer (1972, 1973) (in the kinematical context). Notice also that, as pointed out simultaneously by Niederer (1972, 1973) and Hagen (1972), the subalgebra so(2, 1) deals with non-relativistic 'conformal' coordinate transformations. The above semidirect sum evidently shows the interdependence of *both* subalgebras, a very interesting property (Niederer 1972, 1973) that we call the 'Niederer property'. It is also stressed by the fact that each generator of so(2, 1) can be expressed in terms of those of h(2) so that *all* the generators of the semidirect sum are constructed from bosonic annihilation and creation operators (see (2.2) and (2.3) below).

Recent results (Beckers and Hussin 1986, Beckers *et al* 1987, Dehin and Hussin 1987) in supersymmetric quantum mechanics having already been published by exploiting the above 'Niederer property'; we now want to apply it to coherent states (Klauder and Skagerstam 1985) for the harmonic oscillator. Indeed, in such a field, the subalgebra

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h(2), and evidently the corresponding group, are at the origin of the construction of the so-called ordinary coherent states (Glauber 1963, Klauder 1963) while the subalgebra so(2, 1) and its associated orthogonal group are amongst the first Lie structures to lead to the so-called *generalised* coherent states (Barut and Girardello 1971, Perelomov 1972, 1977, 1986).

Here we want to construct a new set of generalised coherent states for the Heisenberg group. In fact we want to exploit the 'Niederer property' which is particularly useful at the level of the algebras associated with the maximal kinematical invariance group.

Let us first consider the orthogonal subalgebra so(2, 1). It is well known (Wybourne 1974) that in the context of the harmonic oscillator, the eigenvalue of its Casimir operator is $k(k-1)=-\frac{3}{16}$ so that k is either $\frac{1}{4}$ or $\frac{3}{4}$, the value $k=\frac{3}{4}$ leading to a definite positive measure for the Perelomov states (Perelomov 1972, 1977). Secondly, due to the interdependence of so(2, 1) and h(2) through the 'Niederer property', let us choose the Perelomov fundamental state $|k, k\rangle$ with $k=\frac{3}{4}$ for studying the set of Heisenberg generalised coherent states. Then we will show that from this fundamental state $|\frac{3}{4}, \frac{3}{4}\rangle$, we get a new set of normalised Heisenberg states insuring the expected overcompleteness (Klauder and Skagerstam 1985). These states will be characterised by a new normalisation factor and a new measure of specific interest in connection with parasupersymmetric coherent states (Beckers and Debergh 1989b).

As a further comment, let us notice that the choice of the $|k, k\rangle$-state instead of the vacuum state $|0\rangle$ is allowed due to the fact that the respective stationary subalgebras (Perelomov 1979) are identical. In particular, this can also be illustrated for the other value $k=\frac{1}{4}$ with the fundamental state $|\frac{1}{4}, \frac{1}{4}\rangle$ leading to the expected and well known results (Klauder and Skagerstam 1985, Glauber 1963, Klauder 1963, Perelomov 1972, 1977) as mentioned in (3.17).

This paper is organised as follows. In section 2, after a few generalities about the algebras we are interested in, we will put in evidence the different actions of some so(2, 1)- and h(2)-generators on the states we have to consider here. This discussion will lead us to recover the so(2, 1)-Perelomov results. Section 3 will be devoted to the study of the corresponding states in relation with the Heisenberg algebra h(2). This last section will be essentially based on the discovery of the new normalisation factor and the new measure for $k=\frac{3}{4}$.

2. The ($k=\frac{3}{4}$) context and its implications for so(2, 1)

As already mentioned in section 1, the Heisenberg algebra h(2) has played an important role in the first developments on coherent states (Klauder 1963, Glauber 1963, Klauder and Skagerstam 1985). It is generated by three operators (P_+, P_-, I) which satisfy the only non-vanishing commutation relation

$$[P_+, P_-]=-I. \tag{2.1}$$

Through Niederer's realisation (Niederer 1972, 1973) of these generators in terms of creation (a^+) and annihilation (a) operators, we can write them as

$$P_+=\mathrm{i}a^+ \qquad P_-=-\mathrm{i}a. \tag{2.2}$$

The generators of so(2, 1) have also been considered by Niederer; they are expressed as

$$K_+=-(\mathrm{i}/2)(a^+a^+) \qquad K_-=(\mathrm{i}/2)(aa) \qquad K_0=\tfrac{1}{2}(a^+a+\tfrac{1}{2}) \tag{2.3}$$

and satisfy the following commutation relations:

$$[K_0, K_+] = K_+ \qquad [K_0, K_-] = -K_- \qquad [K_+, K_-] = -2K_0. \tag{2.4}$$

The Casimir operator of this simple algebra is given by

$$C_2^{so(2,1)} = K_0^2 - \tfrac{1}{2}[K_+K_- + K_-K_+] \tag{2.5}$$

and is in fact, through our realisation for the harmonic oscillator, identical to $-(\frac{3}{16})I$ as expected (Wybourne 1974). The two irreducible representations of so(2, 1) in the discrete series correspond to $k=\frac{1}{4}$ and $k=\frac{3}{4}$. If we label the states with k and m (the eigenvalue of K_0) we know that (Wybourne 1974, Balantekin *et al* 1988)

$$K_\pm|k, m\rangle = \mp i[(m \pm k)(m \mp k \pm 1)]^{1/2}|k, m \pm 1\rangle \tag{2.6}$$

$$K_\pm|k-\tfrac{1}{2}, m\rangle = \mp i[(m \pm k \mp \tfrac{1}{2})(m \mp k \pm \tfrac{3}{2})]^{1/2}|k-\tfrac{1}{2}, m \pm 1\rangle \tag{2.7}$$

in connection with our choice $k=\frac{3}{4}$. Let us now exploit the work of Niederer (1972, 1973) and simply notice that through the explicit realisation of the generators we are considering, we have

$$P_+P_+ = -2iK_+ \qquad P_-P_- = 2iK_-$$

so that it is possible to define consistently with (2.6) and (2.7) the action of the ladder operators of h(2) as follows:

$$P_\pm|k, m\rangle = i\sqrt{2}(m \mp k \pm 1)^{1/2}|k-\tfrac{1}{2}, m \pm \tfrac{1}{2}\rangle \tag{2.8}$$

$$P_\pm|k-\tfrac{1}{2}, m\rangle = i\sqrt{2}(m \pm k \mp \tfrac{1}{2})^{1/2}|k, m \pm \tfrac{1}{2}\rangle. \tag{2.9}$$

Let us now consider the non-compact algebra so(2, 1). It admits coherent states which can be seen as displacement states defined by

$$|\alpha\rangle = N\, e^{\alpha K_+}|k, k\rangle \tag{2.10}$$

where α is any complex number such that $|\alpha|<1$ (Perelomov 1972, 1977). We have to calculate the normalisation factor N so that we can insure the scalar product $\langle\alpha|\alpha\rangle$ to be equal to 1. By using the explicit action of K_+ on the states $|k, m\rangle$, we get

$$|\alpha\rangle = N \sum_{p=0}^{\infty} (-i\alpha)^p \left(\frac{\Gamma(2k+p)}{\Gamma(p+1)\Gamma(2k)}\right)^{1/2} |k, k+p\rangle. \tag{2.11}$$

Due to the orthonormalisation of the states $|k, m\rangle$, the inner product we have to consider in order to fix N is thus

$$\langle\alpha|\alpha\rangle = |N|^2 \sum_{p=0}^{\infty} \frac{|\alpha|^{2p}\Gamma(2k+p)}{\Gamma(p+1)\Gamma(2k)} = |N|^2(1-|\alpha|^2)^{-2k} \tag{2.12}$$

so that we can conclude that in the case of $k=\frac{3}{4}$, we obtain, according to Perelomov (1972, 1977),

$$|N|^2 = (1-|\alpha|^2)^{3/2}. \tag{2.13}$$

Now, the second requirement (besides the normalisation) for our states to be coherent (Klauder and Skagerstam 1985) is the discovery of a measure $\mu(|\alpha|^2)$ such that the completeness relation is satisfied, i.e.

$$\int |\alpha\rangle\langle\alpha|\mu(|\alpha|^2)\, d^2\alpha = 1 \tag{2.14}$$

where $d^2\alpha = d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha)$.

By combining this relation with (2.11), we are led to

$$\mu(|\alpha|^2)=\frac{1}{2\pi(1-|\alpha|^2)^2}. \tag{2.15}$$

This result coincides with Perelomov's measure when $k=\frac{3}{4}$, while we recall there does not exist such a well defined measure when $k=\frac{1}{4}$ in this so(2, 1) context (Perelomov 1972, 1977).

3. The algebra h(2) and its ($k=\frac{3}{4}$) coherent states

The Heisenberg algebra is a non-compact, non-semisimple, but solvable and nilpotent algebra. This allows us to apply Perelomov's method (Perelomov 1979) by defining our states as

$$|\beta\rangle=N'\,\mathrm{e}^{\beta P_+}|k,k\rangle=N'\sum_{q=0}^{\infty}\frac{\beta^q}{q!}(P_+)^q|k,k\rangle \tag{3.1}$$

where β is any non-vanishing complex number.

By considering the different actions (2.8) and (2.9) of P_+, we have to distinguish the cases where q is even ($q=2s$) or odd ($q=2s+1$) so that we get

$$|\beta\rangle=N'\left[\sum_{s=0}^{\infty}\frac{(\mathrm{i}\beta)^{2s}}{\Gamma(2s+1)}2^s\left(\frac{\Gamma(\frac{3}{2}+s)\Gamma(s+1)}{\Gamma(\frac{3}{2})}\right)^{1/2}|\tfrac{3}{4},\tfrac{3}{4}+s\rangle\right.$$
$$\left.+\sum_{s=0}^{\infty}\frac{(\mathrm{i}\beta)^{2s+1}}{\Gamma(2s+2)}2^s\sqrt{2}\left(\frac{\Gamma(\frac{3}{2}+s)\Gamma(s+2)}{\Gamma(\frac{3}{2})}\right)^{1/2}|\tfrac{1}{4},\tfrac{5}{4}+s\rangle\right] \tag{3.2}$$

where we have replaced k by its chosen value $\frac{3}{4}$. By recalling the well known relation between gamma functions (Abramowitz and Stegun 1965)

$$\Gamma(s+\tfrac{3}{2})2^{2s+2}\Gamma(s+2)=\Gamma(2s+3)\Gamma(\tfrac{1}{2}) \tag{3.3}$$

we can rewrite (3.2) in the form

$$|\beta\rangle=N'\left[\sum_{s=0}^{\infty}(\mathrm{i}\beta)^{2s}\left(\frac{2s+1}{(2s)!}\right)^{1/2}|\tfrac{3}{4},\tfrac{3}{4}+s\rangle+\sum_{s=0}^{\infty}(\mathrm{i}\beta)^{2s+1}\left(\frac{2s+2}{(2s+1)!}\right)^{1/2}|\tfrac{1}{4},\tfrac{5}{4}+s\rangle\right]. \tag{3.4}$$

In order to calculate the normalisation factor N', we again consider the scalar product $\langle\beta|\beta\rangle$; we obviously get

$$|N'|^2=\exp(-|\beta|^2)(1+|\beta|^2)^{-1} \tag{3.5}$$

so that we are able to search for the measure $\mu'(|\beta|^2)$ which satisfies

$$\int|\beta\rangle\langle\beta|\mu'(|\beta|^2)\mathrm{d}^2\beta=1. \tag{3.6}$$

This relation can be put in the form

$$\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{3}{2})\Gamma(s+1)}{[\Gamma(2s+1)]^2\Gamma(\frac{3}{2})}2^{2s}\int\exp(-|\beta|^2)|\beta|^{4s}(1+|\beta|^2)^{-1}\mu'(|\beta|^2)\,\mathrm{d}^2\beta|\tfrac{3}{4},s+\tfrac{3}{4}\rangle\langle\tfrac{3}{4},s+\tfrac{3}{4}|$$
$$+\sum_{s=0}^{\infty}\frac{\Gamma(s+\frac{3}{2})\Gamma(s+2)}{[\Gamma(2s+2)]^2\Gamma(\frac{3}{2})}2^{2s+1}\int\exp(-|\beta|^2)|\beta|^{4s+2}(1+|\beta|^2)^{-1}\mu'(|\beta|^2)\,\mathrm{d}^2\beta$$
$$\times|\tfrac{1}{4},s+\tfrac{5}{4}\rangle\langle\tfrac{1}{4},s+\tfrac{5}{4}|=1. \tag{3.7}$$

By once again using (3.3), we have in fact to find $\mu'(|\beta|^2)$ such that

$$\int \exp(-|\beta|^2)|\beta|^{2q}(1+|\beta|^2)^{-1}\mu'(|\beta|^2)\,d^2\beta=\frac{[\Gamma(q+1)]^2}{\Gamma(q+2)}. \tag{3.8}$$

From this relation, we deduce that

$$\int_0^{+\infty} \exp(-|\beta|^2)\exp(i|\beta|^2 y)(1+|\beta|^2)^{-1}\mu'(|\beta|^2)\,d|\beta|^2=\frac{1}{\pi}\sum_{q=0}^{\infty}\frac{(iy)^q}{(q+1)}. \tag{3.9}$$

This last series absolutely converges when $|y|<1$ and has to be analytically continued otherwise so that we are able to consider

$$M(y)=\frac{1}{\pi}\sum_{q=0}^{\infty}\frac{(iy)^q}{(q+1)}=\frac{1}{\pi}\,{}_2F_1(1,1;2;iy) \tag{3.10}$$

where the hypergeometric function (Abramowitz and Stegun 1965) ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{3.11}$$

Through Fourier transforms (Bateman 1954), the measure we are interested in is thus given by

$$\mu'(|\beta|^2)=\frac{(1+|\beta|^2)}{2\pi^2}\exp(|\beta|^2)\int_{-\infty}^{+\infty}\exp(-i|\beta|^2 y)\,{}_2F_1(1,1;2;iy)\,dy. \tag{3.12}$$

Let us now consider the following change of variables:

$$y=-i(\tfrac{1}{2}-p)$$

so that the measure becomes

$$\mu'(|\beta|^2)=\frac{(1+|\beta|^2)}{2\pi^2}(-i)\exp(|\beta|^2/2)\int_{1/2-i\infty}^{1/2+i\infty}\exp(|\beta|^2 p)\,{}_2F_1(1,1;2;\tfrac{1}{2}-p)\,dp. \tag{3.13}$$

The last integral is in fact the inverse of a Laplace transform and is equal to (Bateman 1954)

$$2\pi i(|\beta|^2)^{-1/2}W_{-1/2,0}(|\beta|^2)$$

where $W_{\lambda,\mu}(x)$ is the Whittaker function (Abramowitz and Stegun 1965). We finally get

$$\mu'(|\beta|^2)=(1/\pi)(1+|\beta|^2)|\beta|^{-1}\exp(|\beta|^2/2)\,W_{-1/2,0}(|\beta|^2). \tag{3.14}$$

Let us mention a few remarks and first notice that the measure (3.14) can also be expressed as

$$\mu'(|\beta|^2)=(1/\pi)(1+|\beta|^2)\exp(|\beta|^2)E_1(|\beta|^2) \tag{3.15}$$

where the exponential integral $E_1(|\beta|^2)$ is proved to satisfy (Abramowitz and Stegun 1965)

$$\exp(|\beta|^2)E_1(|\beta|^2)>1/(1+|\beta|^2). \tag{3.16}$$

As a second remark, let us mention that the parallel study with $k=\frac{1}{4}$ (instead of $\frac{3}{4}$) is evidently easy to realise but starting with well adapted (2.6)-(2.9). The corresponding results would be

$$|N'|^2=\exp(-|\beta|^2) \qquad \mu'(|\beta|^2)=1/\pi \tag{3.17}$$

according to already known developments (Klauder and Skagerstam 1985, Glauber 1963, Klauder 1963). It is then easy to show that

$$\mu'_{(k=3/4)} > \mu'_{(k=1/4)} \tag{3.18}$$

so that we find once again that the $(k=\frac{3}{4})$ context is quite different from the usual $(k=\frac{1}{4})$ context.

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